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THEORETICAL FRAMEWORK FOR MODELLING THE BEHAVIOUR OF FRICTIONAL MATERIALS

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Abstract—A constitutive theory is proposed, which possesses the possibilities of modelling all the important features of the behaviour of frictional materials such as: influence of all three stress invariants, coupling between deviatoric and volumetric response, dilatancy, softening, and different behaviour in loading and unloading. The basic constitutive assumptions are relations between properly defined stress and strain rate invariants, from which the component equations are derived by means of a suitable reformulation. After the incremental stress–strain relations have been derived, they are augmented by consistent loading/unloading criteria. Emphasis is given to a fundamental discussion of the general properties of the theory proposed and it is shown to fulfil all the formal requirements (causality, determinism, admissibility, form-invariance, continuity) that a properly formulated constitutive theory must obey. Moreover, the theory contains a surprisingly large number of classical as well as nonclassical theories as special cases. In particular, it contains formulations ranging from nonassociated plasticity theory, associated plasticity theory, hypoelasticity to elastic-fracturing theory.

INTRODUCTION

The construction of constitutive theories applicable for modelling the time-independent behaviour of frictional materials like concrete, rock and soil represents an intriguing problem, if all essential characteristics of the behaviour should be captured for general nonproportional load histories including unloading. As a consequence, quite different theoretical bases have been applied in the past in an effort to model the material behaviour, e.g. nonlinear elasticity, hypoelasticity, plasticity, and endochronic theory.

In particular, plasticity models have been adopted when the unloading behaviour is of importance. It seems, however, that the plasticity theory, by placing emphasis on the yield surface, focuses on a feature that is only of secondary importance for frictional materials which lack a well-defined yield surface. Moreover, as their theoretical basis is the change and movement of complicated yield/potential surfaces and knowledge of the normal to these surfaces, plasticity models tend to become quite elaborate and complex.

The starting point for the constitutive theory proposed in this paper is completely different. First of all, we start with constitutive assumptions which reflect the incremental stress–strain behaviour directly. Secondly, to be able to identify as simply as possible the material functions involved, these constitutive assumptions relate appropriate stress rate invariants and strain rate invariants. Thirdly, from these invariant expressions we are able to obtain corresponding equations relating the components of the stress rate and strain rate tensors. Then, finally, by taking advantage of the invariant expressions, these incremental stress–strain relations are augmented by consistent loading/unloading criteria.

This paper is devoted to the derivation of this new constitutive theory. No attempt is made to try to calibrate the theory to a specific material, but rather the general properties and potential of the theory are investigated in detail with the time-independent behaviour of frictional materials in mind. It turns out that all essential features of frictional material behaviour can be captured within a rather simple concept and it is of considerable interest that classical formulations ranging from nonassociated plasticity, associated plasticity, hypoelasticity, and elastic-fracturing theory are contained in the theory proposed.

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CONSTITUTIVE MODEL AT INVARIANT LEVEL

One of the essential points in the present formulation is that we start directly with a relation between rates of stress invariants and rates of strain invariants. The following quite general constitutive relation is postulated to hold for a material, which initially is isotropic

$$\begin{bmatrix} \dot{\sigma}_0 \\ \dot{\tau}_0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}_0 \\ \dot{e} \\ \dot{w} \end{bmatrix} \tag{1}$$

where the rate is denoted by means of a dot, and A_1, A_2, A_3, B_1, B_2, and B_3 are material moduli that vary with the loading; all having the dimension of stress.

Employing usual tensor notation and letting σ_{ij} denote the stress tensor, the octahedral normal stress and shear stress are defined by

$$\sigma_0 = \tfrac{1}{3}\sigma_{ii} \tag{2}$$

$$\tau_0^2 = \tfrac{1}{3}s_{ij}s_{ij}, \qquad \text{i.e. } \dot{\tau}_0 = \frac{s_{ij}\dot{s}_{ij}}{3\tau_0} \tag{3}$$

where the deviatoric stress tensor is defined by

$$s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_0. \tag{4}$$

Letting ε_{ij} denote the strain tensor, the octahedral normal strain is defined by

$$\varepsilon_0 = \tfrac{1}{3}\varepsilon_{ii}. \tag{5}$$

In addition, we define the following two strain rate invariants by

$$\dot{e} = \frac{s_{ij}\dot{e}_{ij}}{3\tau_0} \tag{6}$$

$$\dot{w} = \frac{s_{km}s_{ml}\dot{e}_{lk}}{3\tau_0^2} \tag{7}$$

where the deviatoric strain tensor is defined by

$$e_{ij} = \varepsilon_{ij} - \delta_{ij}\varepsilon_0 \tag{8}$$

and where the use of $\dot{e}$ given by eqn (6) was introduced by Resende and Martin[1].

It appears that the two strain rate invariants defined by eqns (6) and (7) bear similarities to the second stress invariant τ_0 and the third stress invariant J_3, respectively, where J_3 is defined by

$$J_3 = \tfrac{1}{3}s_{jk}s_{kl}s_{lj}. \tag{9}$$

Tensile stress and elongation are considered positive quantities and, for convenience, we shall restrict ourselves to small strains and displacements.

The postulated constitutive relation, eqn (1), becomes more apparent when some simplified material models are considered. For linear elasticity we have $A_1 = 3K$, $B_2 = 2G$ and $A_2 = A_3 = B_1 = B_3 = 0$, where K and G denotes the bulk and shear modulus, respectively. Putting $B_1 = B_3 = A_3 = 0$ we obtain the invariant formulation of the elastic-fracturing theory of Resende and Martin[1]. In concrete literature, nonlinear models based on an octahedral formulation are often applied. In this case, cf. Gerstle[2], Scavuzzo *et al.*[3] and Stankowski and Gerstle[4], the octahedral shear strain rate $\dot{\gamma}_0$ is used instead of $\dot{e}$

and the influence of $\dot{w}$ is disregarded. Noting that $\dot{e}$ and $\dot{\gamma}_0$ become identical for loading in the Rendulic plane, in which two of the principal stresses are equal, the present invariant relations bear similarities to these octahedral formulations. Moduli A_2 and B_1 can therefore be interpreted as moduli reflecting the coupling between deviatoric loading and volumetric response (and vice versa), characteristic for the behaviour of frictional materials. It will be shown later that moduli A_3 and B_3 represent the influence of the direction of the deviatoric loading, i.e. the third stress invariant.

CONSTITUTIVE MODEL AT COMPONENT LEVEL

Equation (1) defines the constitutive model in terms of invariants. It is obvious, however, that if the model is to be of any use for general stress paths, equations must be derived for all the stress and strain components that are in accordance with eqn (1). Whereas this expression consists of two equations, six equations are needed to prescribe the relations between the stress and strain components, so this step in the constitutive formulation is by no means evident. However, it can be achieved by a suitable reformulation of the invariant equations.

Modulus B_2 is split into two parts, i.e.

$$B_2 = B_2^e + B_2^c \tag{10}$$

where B_2^e can be considered as the elastic part corresponding to $2G$ and B_2^c represents a correction term due to the nonlinear behaviour. Multiplying eqns $(1)_2$ by τ_0 and using eqn (10) we obtain

$$\tau_0\dot{\tau}_0 = B_2^e\dot{e}\tau_0 + (B_1\dot{\varepsilon}_0 + B_2^c\dot{e} + B_3\dot{w})\tau_0. \tag{11}$$

The different terms in this expression shall now be reformulated. Using eqns (3) and (6) we have

$$\tau_0\dot{\tau}_0 = \tfrac{1}{3}s_{ij}\dot{s}_{ij}; \qquad \dot{e}\tau_0 = \tfrac{1}{3}s_{ij}\dot{e}_{ij}; \qquad \tau_0 = \frac{\tau_0^2}{\tau_0} = \frac{1}{3\tau_0}s_{ij}s_{ij}. \tag{12}$$

Using eqn (12) in eqn (11) yields

$$s_{ij}\left(\dot{s}_{ij} - B_2^e\dot{e}_{ij} - \frac{B_1\dot{\varepsilon}_0 + B_2^c\dot{e} + B_3\dot{w}}{\tau_0}s_{ij}\right) = 0. \tag{13}$$

Excluding the trivial case where $s_{ij} = 0$, two possibilities exist for fulfilling eqn (13). Noting that s_{ij} and the expression in parentheses can be interpreted as vectors, eqn (13) is satisfied if the two vectors are normal to each other, or if the components of the expression in parentheses are equal to zero. Equation (13) must hold in general, i.e. also for linear elasticity for which $\dot{s}_{ij} = B_2^e\dot{e}_{ij}$ and the last term in parentheses is zero. Therefore, in this case we cannot use the solution possibility that the two vectors are normal to each other. Consequently, to satisfy eqn (13) for arbitrary loadings we are left with the other possibility, namely that all the components of the expression in parentheses are equal to zero, i.e.

$$\dot{s}_{ij} = B_2^e\dot{e}_{ij} + \frac{B_1\dot{\varepsilon}_0 + B_2^c\dot{e} + B_3\dot{w}}{\tau_0}s_{ij}. \tag{14}$$

Consequently, the invariant eqn (1) has been reformulated so that we have derived the component eqn (14). By multiplying eqn (14) by s_{ij} the invariant eqn (1) follows directly. However, even though the derivation of eqn (14) seems quite natural, it should be noted that there does not exist a one-to-one relation between the invariant equation and the component equations. In fact, different component equations can be envisaged all resulting in the same invariant equation. On the other hand, out of this spectrum of component equations we should choose a system which is in accordance with general theorems and experiences within the field of constitutive modelling. It will be shown later that eqn (14) fulfils such requirements. Note in the first place that as B_2^e is equivalent to $2G$, the first term on the right-hand side of eqn (14) corresponds to the elastic response, whereas the second term causes the nonlinear behaviour.

Equation (14) determines the deviatoric response. The volumetric response is still given by eqn (1); however, similarly to eqn (10) we split the modulus A_1 into two parts, i.e.

$$A_1 = A_1^e + A_1^c \tag{15}$$

where A_1^e can be considered as the elastic part corresponding to $3K$ and A_1^c represents a correction term due to the nonlinear behaviour. Therefore, the volumetric response is controlled by

$$\dot{\sigma}_0 = A_1^e \dot{\varepsilon}_0 + (A_1^c \dot{\varepsilon}_0 + A_2 \dot{e} + A_3 \dot{w}) \tag{16}$$

where the first term on the right-hand side corresponds to the elastic response, whereas the second term causes the nonlinear behaviour.

Disregarding for the time being differences in the loading and unloading behaviour, eqns (14) and (16) determine the stress–strain behaviour completely.

CONSISTENT LOADING/UNLOADING CRITERIA

As the aim is to formulate a theory applicable to general loadings, we must augment the stress–strain relations derived with criteria that distinguish loading and unloading. The introduction of a loading/unloading criterion infers in general that different constitutive equations are applied in the loading and unloading regions and that neutral loading separates these regions. In this paper, it is required that an adopted loading/unloading criteron must be consistent in the sense of Handelman *et al.*[5], implying that these constitutive equations must be identical for neutral loading. In accordance with general expectations, this requirement ensures that the response varies continuously for different imposed loadings. With a view to later numerical applications, this continuity requirement is also of essential importance for the establishment of a stable, convergent numerical scheme.

It is of significance that in frictional materials, nonlinearity can be caused by two different types of loadings: volumetric or deviatoric loading. This suggests the introduction of two corresponding loading/unloading criteria. In addition, these criteria should be applicable to the deviatoric response, eqn (14), as well as to the volumetric response, eqn (16). As the first term in each of these expressions corresponds to elastic behaviour, it seems natural to relate the deviatoric and volumetric loading criteria to the remaining terms.

To achieve this objective we note that it is always possible to write the moduli appearing in the nonlinear terms as $A_1^c = A_1'^c + A_1''^c$, $B_1 = B_1' + B_1''$, etc. Moreover, if it is assumed that $B_1'' = p_1 A_1'^c$, $A_1''^c = p_2 B_1'$, etc. then the following decompositions result:

$$A_1^c = A_1'^c + p_2 B_1'; \qquad A_2 = A_2' + p_2 B_2'^c; \qquad A_3 = A_3' + p_2 B_3' \tag{17}$$

$$B_1 = B_1' + p_1 A_1'^c; \qquad B_2^c = B_2'^c + p_1 A_2'; \qquad B_3 = B_3' + p_1 A_3' \tag{18}$$

where p_1 and p_2 are dimensionless functions, i.e. instead of the six nonlinear moduli A_1, A_2, A_3, B_1, B_2, and B_3 introduced initially we now deal with eight independent functions $A_1'^c$, A_2', A_3', B_1', $B_2'^c$, B_3', p_1, and p_2.

Let us now define the "deviatoric" loading function by

$$\dot{L}_d = -(B_1' \dot{\varepsilon}_0 + B_2'^c \dot{e} + B_3' \dot{w}); \qquad L_d = -\left(\int B_1' \, d\varepsilon_0 + \int B_2'^c \, de + \int B_3' \, dw \right) \tag{19}$$

and the "volumetric" loading function by

$$\dot{L}_v = A_1'^c\dot{\varepsilon}_0 + A_2'\dot{e} + A_3'\dot{w}; \qquad L_v = \int A_1'^c\,\mathrm{d}\varepsilon_0 + \int A_2'\,\mathrm{d}e + \int A_3'\,\mathrm{d}w. \tag{20}$$

Using eqns (19) and (20) in the constitutive eqns (14) and (16), we obtain

$$\dot{e}_{ij} = \frac{\dot{s}_{ij}}{B_2^c} + \frac{\dot{L}_d - p_1\dot{L}_v}{B_2^c\tau_0}s_{ij} \tag{21}$$

$$\dot{\varepsilon}_0 = \frac{\dot{\sigma}_0}{A_1^c} - \frac{\dot{L}_v - p_2\dot{L}_d}{A_1^c}. \tag{22}$$

It is assumed that so-called deviatoric unloading occurs when $\dot{L}_d \leqslant 0$ imply that in the stress–strain relations we apply $B_1' = B_2'^c = B_3' = 0$. For deviatoric loading, $\dot{L}_d \geqslant 0$ applies and the moduli B_1', $B_2'^c$, and B_3' can take arbitrary values in the constitutive equations. Likewise, it is assumed that so-called volumetric unloading occurs when $\dot{L}_v \leqslant 0$ implying that in the stress–strain relations we apply $A_1'^c = A_2' = A_3' = 0$. For volumetric loading, $\dot{L}_v \geqslant 0$ applies and the moduli $A_1'^c$, A_2', and A_3' can take arbitrary values in the constitutive equations.

With the loading/unloading criteria suggested above, the stress–strain relations in the loading and unloading regimes become identical during any neutral loading thereby ensuring the continuity of the response. Thus, the constitutive equations given by eqns (14) and (16) have been augmented by consistent loading/unloading criteria.

Note that A_1^c and B_2^c need not be constant moduli implying that the so-called elastoplastic coupling, i.e. reduction of unloading stiffness, characteristic for severely loaded frictional materials, cf. Ref. [6], can be modelled.

It is of importance to observe that the loading/unloading criteria introduced have been formulated in terms of strain rate quantities. Therefore, these criteria can be applied even within a strain softening formulation. Moreover, it will be shown later that the loading criteria can be interpreted also in the usual stress space inferring that neutral loadings given by $\dot{L}_v = 0$ and $\dot{L}_d = 0$, in general, correspond to two different surfaces in the stress space. In that respect, it is important to note that according to eqns (21) and (22), nonlinear deviatoric and volumetric response follows if just one of the loading functions is activated.

TANGENTIAL STIFFNESS TENSOR

The material behaviour is determined by the component eqns (14) and (16). For convenience, these expressions shall now be rewritten so that the tangential stiffness tensor and thereby the tangential stiffness matrix, which is of importance in numerical applications, is given explicitly.

Noting eqns (4), (14) and (16) can be combined to

$$\dot{\sigma}_{ij} = B_2^c\dot{e}_{ij} + \left[\frac{B_1}{\tau_0}s_{ij} + (A_1^c + A_1^c)\delta_{ij}\right]\dot{\varepsilon}_0 + \frac{1}{3}\left(\frac{B_2^c}{\tau_0^2}s_{ij} + \frac{A_2}{\tau_0}\delta_{ij}\right)s_{kl}\dot{e}_{kl}$$
$$+\frac{1}{3}\left(\frac{B_3}{\tau_0^3}s_{ij} + \frac{A_3}{\tau_0^2}\delta_{ij}\right)s_{km}s_{ml}\dot{e}_{lk} \tag{23}$$

where use has been made of eqns (6) and (7). Now, by means of eqns (5) and (8) and observing that $s_{kl}\dot{e}_{kl} = s_{kl}\dot{\varepsilon}_{kl}$ and $s_{km}s_{ml}\dot{e}_{lk} = s_{km}s_{ml}\dot{\varepsilon}_{lk} - \tau_0^2\dot{\varepsilon}_{kk}$ we obtain

$$\dot{\sigma}_{ij} = B_2^c\dot{\varepsilon}_{ij} + \frac{1}{3}\left[\frac{B_1 - B_3}{\tau_0}s_{ij} + (A_1^c + A_1^c - B_2^c - A_3)\delta_{ij}\right]\dot{\varepsilon}_{kk}$$
$$+\frac{1}{3}\left(\frac{B_2^c}{\tau_0^2}s_{ij} + \frac{A_2}{\tau_0}\delta_{ij}\right)s_{kl}\dot{\varepsilon}_{kl} + \frac{1}{3}\left(\frac{B_3}{\tau_0^3}s_{ij} + \frac{A_3}{\tau_0^2}\delta_{ij}\right)s_{km}s_{ml}\dot{\varepsilon}_{lk} \tag{24}$$

or

$$\dot{\sigma}_{ij} = D_{ijkl}\dot{\varepsilon}_{kl} \tag{25}$$

where the tangential stiffness tensor is given by

$$D_{ijkl} = \frac{B_2^c}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{1}{3}\left[\frac{B_1 - B_3}{\tau_0} s_{ij} + (A_1^c + A_1^c - B_2^c - A_3)\delta_{ij}\right]\delta_{kl}$$
$$+ \frac{1}{3}\left(\frac{B_2^c}{\tau_0^2} s_{ij} + \frac{A_2}{\tau_0}\delta_{ij}\right)s_{kl} + \frac{1}{3}\left(\frac{B_3}{\tau_0^3} s_{ij} + \frac{A_3}{\tau_0^2}\delta_{ij}\right)s_{km}s_{ml} \tag{26}$$

and where the actual values of the material moduli depend upon the loading/unloading criteria through eqns (17) and (18). With expressions (25) and (26), we have derived a form of the constitutive equations, which directly relates the rate of the stress tensor to the rate of the total strain tensor through the tangential stiffness tensor, which only depends on the current state of material, i.e. we have obtained a linearized form of the incremental constitutive equations. The matrix formulation of eqn (25) directly applicable in numerical schemes is easily obtained from eqn (24).

Let us now investigate the properties of D_{ijkl} given by eqn (26). In accordance with the symmetry of σ_{ij} and ε_{ij}, it appears that the conditions $D_{ijkl} = D_{jikl} = D_{jilk}$ are always fulfilled. To fulfil also the requirement of a symmetric tangential stiffness matrix, which is equivalent to the condition $D_{ijkl} = D_{klij}$, we must impose $A_3 = B_3 = 0$ and $A_2 = B_1$. However, it will be shown later that the inclusion of the material moduli A_3 and B_3 is very important for modelling of frictional materials as these moduli reflect the influence of the third stress invariant.

MATERIAL FORM-INVARIANCE

It is apparent that the constitutive theory proposed obeys all the more obvious axioms like causality, determinism and admissibility related to an adequate theory, cf. Eringen[7]. However, as demonstrated by Bazant[8], the additional requirement of form-invariance is a very important issue to which special attention must be paid, if a proper constitutive theory is to be formulated. The fulfilment of this requirement is proven below for the constitutive theory proposed here.

In general, the form-invariance principle states that the material response is independent of the reference configuration. In the present paper, where we deal only with initially isotropic materials, form-invariance requires that the constitutive equations remain the same irrespective of the coordinate system. For a constitutive relation having the form of eqn (25), this requirement infers that the most general form of the constitutive relation becomes, cf. Refs [9, 10]

$$\dot{\sigma}_{ij} = a_0\delta_{ij}\dot{\varepsilon}_{kk} + a_1\dot{\varepsilon}_{ij} + a_2\sigma_{ij}\dot{\varepsilon}_{kk} + a_3\delta_{ij}\sigma_{mn}\dot{\varepsilon}_{mn} + a_4(\sigma_{im}\dot{\varepsilon}_{mj} + \dot{\varepsilon}_{mi}\sigma_{mj})$$
$$+ a_5\sigma_{im}\sigma_{mj}\dot{\varepsilon}_{kk} + a_6\sigma_{mn}\sigma_{ij}\dot{\varepsilon}_{nm} + a_7\sigma_{mn}\sigma_{nk}\delta_{ij}\dot{\varepsilon}_{km}$$
$$+ a_8(\sigma_{im}\sigma_{mk}\dot{\varepsilon}_{kj} + \dot{\varepsilon}_{im}\sigma_{mk}\sigma_{kj}) + a_9\sigma_{mn}\sigma_{ik}\sigma_{kj}\dot{\varepsilon}_{nm}$$
$$+ a_{10}\sigma_{mn}\sigma_{nk}\sigma_{ij}\dot{\varepsilon}_{km} + a_{11}\sigma_{mn}\sigma_{nk}\sigma_{ir}\sigma_{rj}\dot{\varepsilon}_{km} \tag{27}$$

where the 12 functions $a_0, \ldots, a_{11}$, in general, depend on the invariants. Disregarding any loading/unloading criteria, eqn (27) represents the most general hypoelastic model. A comparison with eqn (24) shows immediately that we must have $a_{11} = 0$, if eqn (24) should is a special case of eqn (27). Using eqn (4), the expression above can then, after some algebra, be rewritten as

$$\begin{aligned}\dot{\sigma}_{ij} = {} & (a_1+2a_4\sigma_0+2a_8\sigma_0)\dot{\varepsilon}_{ij}+[(a_5+a_9\sigma_0)s_{im}s_{mj}+(a_2+2a_5\sigma_0+a_6\sigma_0+2a_9\sigma_0^2+a_{10}\sigma_0^2)s_{ij} \\ & +(a_0+a_2\sigma_0+a_3\sigma_0+a_5\sigma_0^2+a_6\sigma_0^2+a_7\sigma_0^2+a_9\sigma_0^3+a_{10}\sigma_0^3)\delta_{ij}]\dot{\varepsilon}_{kk} \\ & +[(a_6+2a_9\sigma_0+2a_{10}\sigma_0)s_{ij}+(a_3+a_6\sigma_0+2a_7\sigma_0+a_9\sigma_0^2+2a_{10}\sigma_0^2)\delta_{ij}+a_9s_{ik}s_{kj}]s_{mn}\dot{\varepsilon}_{mn} \\ & +[a_{10}s_{ij}+(a_7+a_{10}\sigma_0)\delta_{ij}]s_{km}s_{ml}\dot{\varepsilon}_{lk}+(a_4+2a_8\sigma_0)(s_{im}\dot{\varepsilon}_{mj}+s_{jm}\dot{\varepsilon}_{mi}) \\ & +a_8(s_{im}s_{mk}\dot{\varepsilon}_{kj}+s_{mk}s_{kj}\dot{\varepsilon}_{im}). \end{aligned} \tag{28}$$

We shall now continue to investigate whether eqn (28) contains the formulation given by eqn (24). From the last term of eqn (28) it appears that we must have $a_8 = 0$, which from the second last term, infers that $a_4 = 0$. From the coefficient of $s_{mn}\dot{\varepsilon}_{mn}$, it appears that we must choose $a_9 = 0$. From the term $s_{im}s_{mj}$ present in the coefficient in front of $\dot{\varepsilon}_{kk}$, $a_9 = 0$ infers that $a_5 = 0$. Thus, in conclusion we must choose

$$a_4 = a_5 = a_8 = a_9 = a_{11} = 0 \tag{29}$$

which reduces eqn (28) to

$$\begin{aligned}\dot{\sigma}_{ij} = {} & a_1\dot{\varepsilon}_{ij}+\{(a_2+a_6\sigma_0+a_{10}\sigma_0^2)s_{ij}+[a_0+(a_2+a_3)\sigma_0+(a_6+a_7)\sigma_0^2+a_{10}\sigma_0^3]\delta_{ij}\}\dot{\varepsilon}_{kk} \\ & +\{(a_6+2a_{10}\sigma_0)s_{ij}+[a_3+(a_6+2a_7)\sigma_0+2a_{10}\sigma_0^2]\delta_{ij}\}s_{mn}\dot{\varepsilon}_{mn} \\ & +[a_{10}s_{ij}+(a_7+a_{10}\sigma_0)\delta_{ij}]s_{km}s_{ml}\dot{\varepsilon}_{lk}. \end{aligned} \tag{30}$$

A comparison with eqn (24) shows that if we choose

$$A_1 = A_1^c+A_1^c = a_1+3[a_0+(a_2+a_3)\sigma_0+a_6\sigma_0^2+a_7(\sigma_0^2+\tau_0^2)+a_{10}\sigma_0(\sigma_0^2+\tau_0^2)] \tag{31}$$

$$A_2 = 3[a_3+(a_6+2a_7)\sigma_0+2a_{10}\sigma_0^2]\tau_0 \tag{32}$$

$$A_3 = 3(a_7+a_{10}\sigma_0)\tau_0^2 \tag{33}$$

$$B_1 = 3[a_2+a_6\sigma_0+a_{10}(\sigma_0^2+\tau_0^2)]\tau_0 \tag{34}$$

$$B_2^c = a_1 \tag{35}$$

$$B_2^c = 3(a_6+2a_{10}\sigma_0)\tau_0^2 \tag{36}$$

$$B_3 = 3a_{10}\tau_0^3 \tag{37}$$

then eqn (24) becomes identical with eqn (30), i.e. it has been proven that the proposed constitutive theory is formulated in such a way that it also satisfies the form-invariance requirement. This, in combination with the consistent loading/unloading criteria, means that the theory proposed fulfils all the formal requirements that can be related to a proper constitutive theory. Moreover, the constitutive formulation proposed takes a particularly simple form in terms of invariants, cf. eqn (1), thereby facilitating the identification of the material moduli.

Disregarding the loading/unloading criteria, it appears that the proposed constitutive theory becomes identical to a hypoelastic formulation of the third grade. It is of considerable interest, however, that the theory contains all the zero grade terms, but only some of the first, second and third grade terms. If $A_3 = B_3 = 0$ then eqns (33) and (37) show that the formulation reduces to a second grade hypoelastic model.

CHANGE OF THIRD STRESS INVARIANT-INVERSION OF EQUATIONS

The exposition above contains in principle all information regarding the constitutive theory proposed. However, to be able to investigate the implications of this theory in more detail, it becomes convenient to derive some additional expressions for the constitutive theory at the invariant level.

The rate of the first two stress invariants σ_0 and τ_0 is given directly by eqn (1). An expression for the third stress invariant is given by J_3 defined by eqn (9), from which we obtain $\dot{J}_3 = s_{jk}s_{ki}\dot{s}_{ij}$. Therefore, to obtain an expression for the rate of the third stress invariant, we multiply eqn (14) by $s_{jk}s_{ki}$ and obtain

$$\frac{\dot{J}_3}{3\tau_0^2} = B_1\frac{J_3}{\tau_0^3}\dot{\varepsilon}_0 + B_2^c\frac{J_3}{\tau_0^3}\dot{e} + \left(B_2^c + B_3\frac{J_3}{\tau_0^3}\right)\dot{w} \tag{38}$$

where eqn (7) has been utilized. Defining the moduli C_1, C_2, C_3 by

$$C_1 = B_1\frac{J_3}{\tau_0^3}; \qquad C_2 = B_2^c\frac{J_3}{\tau_0^3}; \qquad C_3 = B_2^c + B_3\frac{J_3}{\tau_0^3} \tag{39}$$

as well as the rate of the invariant stress measure J by

$$\dot{J} = \frac{\dot{J}_3}{3\tau_0^2} \tag{40}$$

we can combine eqns (1) and (38) as follows

$$\begin{bmatrix} \dot{\sigma}_0 \\ \dot{\tau}_0 \\ \dot{J} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}_0 \\ \dot{e} \\ \dot{w} \end{bmatrix}. \tag{41}$$

At the failure surface for the material in question, the stresses take their peak values. From eqn (41) this infers that at failure the determinant of the coefficient matrix must be zero thereby providing a failure condition. The determinant is given by

$$\det = B_2^c\left[A_1B_2 - A_2B_1 + \frac{J_3}{\tau_0^3}(A_1B_3 - A_3B_1)\right] \tag{42}$$

where eqns (39) and (10) have been utilized. Moreover, we shall later need to express the strain rate invariants $\dot{\varepsilon}_0$, $\dot{e}$, and $\dot{w}$ in terms of the stress rate invariants, $\dot{\sigma}_0$, $\dot{\tau}_0$, and $\dot{J}$. This can be accomplished by inverting eqn (41), whereby we obtain

$$\begin{bmatrix} \dot{\varepsilon}_0 \\ \dot{e} \\ \dot{w} \end{bmatrix} = \frac{1}{\det}\begin{bmatrix} B_2C_3 - B_3C_2 & -A_2C_3 + A_3C_2 & A_2B_3 - A_3B_2 \\ -B_1C_3 + B_3C_1 & A_1C_3 - A_3C_1 & -A_1B_3 + A_3B_1 \\ B_1C_2 - B_2C_1 & -A_1C_2 + A_2C_1 & A_1B_2 - A_2B_1 \end{bmatrix} \begin{bmatrix} \dot{\sigma}_0 \\ \dot{\tau}_0 \\ \dot{J} \end{bmatrix}. \tag{43}$$

It appears readily from this expression that by proper choices of the material moduli, we are able to model volume compaction and dilation.

Moreover, the existence of the invariant expression (43) implies that it is simple to invert even the constitutive relation (25) and obtain the tangential compliance tensor explicitly. Using expression (43), the strain rate invariants in eqns (14) and (16) can be eliminated so that $\dot{e}_{ij}$ and $\dot{\varepsilon}_0$ can be expressed in terms of stresses and stress rates. Using eqns (4) and (8), it is then easily shown that

$$\dot{\varepsilon}_{ij} = C_{ijkl}\dot{\sigma}_{kl} \tag{44}$$

where the tangential compliance tensor C_{ijkl} is given by

$$C_{ijkl} = \frac{1}{2B_2^e}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\frac{1}{3\det}\left\{\left[B_2^e\left(B_2+B_3\frac{J_3}{\tau_0^3}\right)-A_2B_3+A_3B_2-\frac{\det}{B_2^e}\right]\delta_{ij}\right.$$
$$\left.-(A_3B_1-A_1B_3+B_1B_2^e)\frac{s_{ij}}{\tau_0}\right\}\delta_{kl}-\frac{1}{3\tau_0\det}\left\{(A_1B_2^e-A_2B_1)\frac{s_{ij}}{\tau_0}\right.$$
$$\left.+\left[A_2B_2^e+\frac{J_3}{\tau_0^3}(A_2B_3-A_3B_2^e)\right]\delta_{ij}\right\}s_{kl}-\frac{1}{3\tau_0^2\det}\left\{(A_1B_3-A_3B_1)\frac{s_{ij}}{\tau_0}\right.$$
$$\left.+(A_3B_2-A_2B_3)\delta_{ij}\right\}s_{km}s_{lm}. \tag{45}$$

REFORMULATION OF THE LOADING FUNCTIONS IN STRESS SPACE

The loading functions $\dot{L}_d$ and $\dot{L}_v$ were expressed previously in terms of strain rate quantities, cf. eqns (19) and (20), which, in fact, is very convenient because it makes possible consideration of strain softening. However, to obtain a physical interpretation of the loading functions, it is advantageous to apply stress rates instead of strain rates, as this permits interpretation in the usual stress space.

For this purpose we observe that expression (43) together with eqns (17), (18), and (39) provide expressions for $\dot{\varepsilon}_0$, $\dot{e}$ and $\dot{w}$. Using these expressions in the loading functions given by eqns (19) and (20), we obtain after lengthy, but trivial calculations that the deviatoric loading function $\dot{L}_d$ becomes

$$\dot{L}_d = -\frac{1}{\det}\left\{B_2^e\left[B_2^eB_1'+p_1(A_2'B_1'-A_1'^eB_2'^e)+p_1(A_3'B_1'-A_1'^eB_3')\frac{J_3}{\tau_0^3}\right]\dot{\sigma}_0\right.$$
$$+\left[B_2^e(A_1^eB_2'^e+A_1'^eB_2'^e-A_2'B_1')+p_1A_1^e(A_3'B_2'^e-A_2'B_3')\frac{J_3}{\tau_0^3}\right]\dot{\tau}_0$$
$$\left.+[B_2^e(A_1'^eB_3'-A_3'B_1'+A_1^eB_3')+p_1A_1^e(A_2'B_3'-A_3'B_2'^e)]\dot{J}\right\}. \tag{46}$$

Likewise, the volumetric loading function $\dot{L}_v$ is given by

$$\dot{L}_v = \frac{1}{\det}\left\{B_2^e\left[A_1'^eB_2^e+A_1'^eB_2'^e-A_2'B_1'+(A_1'^eB_3-A_3'B_1')\frac{J_3}{\tau_0^3}\right]\dot{\sigma}_0\right.$$
$$+\left[A_1^eA_2'B_2^e+p_2B_2^e(A_2'B_1'-A_1'^eB_2'^e)+A_1^e(A_2'B_3'-A_3'B_2'^e)\frac{J_3}{\tau_0^3}\right]\dot{\tau}_0$$
$$\left.+[A_1^e(A_3'B_2^e-A_2'B_3'+A_3'B_2'^e)+p_2B_2^e(A_3'B_1'-A_1'^eB_3')]\dot{J}\right\}. \tag{47}$$

Whereas the influence of moduli A_1^e and B_2^e is readily interpreted and moduli A_2 and B_1 are responsible for the coupling between deviatoric and volumetric response, we are now in a position to evaluate the influence of moduli A_3 and B_3. Suppose we have neutral deviatoric loading, i.e. $\dot{L}_d = 0$. Furthermore, suppose that the loading is purely deviatoric, i.e. $\dot{\sigma}_0 = 0$. From eqn (46) we then obtain

$$\left[B_2^e(A_1^eB_2'^e+A_1'^eB_2'^e-A_2'B_1')+p_1A_1^e(A_3'B_2'^e-A_2'B_3')\frac{J_3}{\tau_0^3}\right]\dot{\tau}_0$$
$$+[B_2^e(A_1'^eB_3'-A_3'B_1'+A_1^eB_3')+p_1A_1^e(A_2'B_3'-A_3'B_2'^e)]\dot{J} = 0. \tag{48}$$

It appears that if $A_3' = B_3' = 0$, i.e. $A_3 = B_3 = 0$, then the solution to eqn (48) becomes

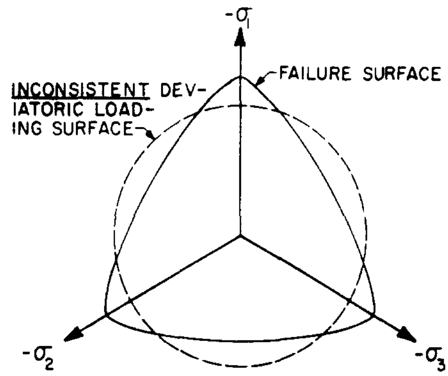

Fig. 1. Deviatoric plane. Inconsistency of failure curve for frictional materials and deviatoric loading curve (= circle) when $A_3 = B_3 = 0$.

$\dot{\tau}_0 = 0$, i.e. the neutral deviatoric loading curve in the deviatoric plane in question becomes a circle. Therefore, only when moduli A_3 and B_3 are different from zero, is it possible to simulate loading curves in the deviatoric plane that depend on the third stress invariant. Consequently, inclusion of moduli A_3 and B_3 is of utmost importance when modelling the behaviour of frictional materials. Similar conclusions can be derived from the expression for the volumetric loading function.

If $A_3 = B_3 = 0$, then eqn (42) provides at failure $A_1B_2 - A_2B_1 = 0$. It appears that by letting these moduli depend on the third stress invariant, we obtain a failure criterion which also depends on the third stress invariant. However, the discussion above shows that for $A_3 = B_3 = 0$, the neutral loading surfaces will not depend on the third stress invariant implying an inconsistency in the sense that the circular deviatoric loading curve might intersect the failure curve. This situation is sketched in Fig. 1, and it emphasizes the requirement for inclusion of moduli A_3 and B_3 when modelling frictional materials.

Let us now pursue the investigation of the surfaces in the stress space corresponding to neutral deviatoric loading, i.e. $\dot{L}_d = 0$, and neutral volumetric loading, i.e. $\dot{L}_v = 0$. These expressions are readily obtained from eqns (46) and (47), but a particular simple form results if we choose

$$A'_2B'_3 = A'_3B'^c_2. \tag{49}$$

This choice infers that neutral deviatoric loading $\dot{L}_d = 0$ becomes

$$\dot{\tau}_0 + \frac{B^c_2B'_1 + p_1(A'_2B'_1 - A'^c_1B'^c_2)\left(1 + \frac{A'_3}{A'_2}\frac{J_3}{\tau_0^3}\right)}{A^c_1B'^c_2 - A'_2B'_1 + A'^c_1B'^c_2}\dot{\sigma}_0 + \frac{A'_3}{A'_2}\dot{J} = 0 \tag{50}$$

and neutral volumetric loading $\dot{L}_v = 0$ corresponds to

$$\dot{\tau}_0 + \frac{A'^c_1B^c_2 - (A'_2B'_1 - A'^c_1B'^c_2)\left(1 + \frac{A'_3}{A'_2}\frac{J_3}{\tau_0^3}\right)}{A^c_1A'_2 + p_2(A'_2B'_1 - A'^c_1B'^c_2)}\dot{\sigma}_0 + \frac{A'_3}{A'_2}\dot{J} = 0. \tag{51}$$

Comparison of eqns (50) and (51) shows that only the coefficient in front of $\dot{\sigma}_0$ differs. This interesting aspect means that the two neutral loading surfaces have the same form in the deviatoric plane and that the surfaces, in general, intersect each other in a plane characterized by $\dot{\sigma}_0 = 0$, as illustrated in Fig. 2.

Now, tests for frictional materials[11–13] demonstrate that the failure surface in the stress space is highly insensitive to the particular stress path. This suggests that it is advantageous, in some way or another, to incorporate the features of the stress failure surface into the constitutive model. One way of achieving this is to choose the material

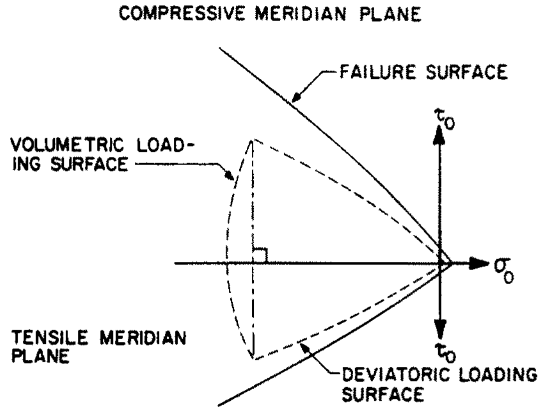

Fig. 2. Appearance of the neutral volumetric surface and the neutral deviatoric surface, which intersect each other in a plane curve located in the deviatoric plane. The neutral deviatoric surface can be made affine to the failure surface, as shown below.

moduli so that the surface, which corresponds to neutral deviatoric loading, becomes affine to the stress failure surface. For a given failure criterion, which is expressed by the three stress invariants, this modulus calibration is easily performed, as demonstrated in the next section.

CALIBRATION TO A FAILURE CRITERION FOR CONCRETE

We shall make use of the following failure criterion, which has been shown to be in close agreement with experimental data for concrete[14, 15]

$$A\frac{J_2}{\sigma_c^2}+\lambda(\theta)\frac{\sqrt{J_2}}{\sigma_c}+B\frac{I_1}{\sigma_c}-1=0 \tag{52}$$

where $J_2 = 3\tau_0^2/2$ and $I_1 = 3\sigma_0$. Moreover

$$\lambda(\theta)=\begin{cases} K_1\cos\left[\frac{1}{3}\arccos\left(K_2\cos 3\theta\right)\right]; & \text{for } \cos 3\theta \geqslant 0 \\ K_1\cos\left[\frac{\pi}{3}-\frac{1}{3}\arccos\left(-K_2\cos 3\theta\right)\right]; & \text{for } \cos 3\theta \leqslant 0 \end{cases} \tag{53}$$

and A, B, K_1, and K_2 are nonnegative dimensionless parameters ($0 \leqslant K_2 \leqslant 1$), whereas σ_c denotes the uniaxial compressive strength value of the concrete ($\sigma_c > 0$). In addition, the invariant $\cos 3\theta$ is defined by

$$\cos 3\theta = \sqrt{2}\frac{J_3}{\tau_0^3}. \tag{54}$$

Note that the dependence of the θ-angle disappears if $K_2 = 0$, which infers that $\lambda = \sqrt{3}K_1/2$. It follows that eqn (52) reduces to the classical criteria of Drucker and Prager[16] for $K_2 = A = 0$ and of von Mises for $K_2 = A = B = 0$.

Now, eqn (52) determines the stress state at failure; however, for any stress state there exists a positive σ_*-value so that the following equation is fulfilled

$$\frac{3A}{2}\left(\frac{\tau_0}{\sigma_*}\right)^2+\sqrt{\left(\frac{3}{2}\right)}\lambda(\theta)\frac{\tau_0}{\sigma_*}+3B\frac{\sigma_0}{\sigma_*}-1=0 \tag{55}$$

where we have used $J_2 = 3\tau_0^2/2$ and $I_1 = 3\sigma_0$. The surface determined by this equation is affine to the failure surface.

In the following we need an expression for $\dot{\lambda}$ and from eqn (53) we obtain

$$\dot{\lambda} = \begin{cases} \dfrac{K_1 K_2}{3} \dfrac{\sin\left[\frac{1}{3}\arccos\left(K_2 \cos 3\theta\right)\right]}{\left(1-K_2^2\cos^2 3\theta\right)^{1/2}} \dfrac{\partial \cos 3\theta}{\partial t}; & \text{for } \cos 3\theta \geqslant 0 \\ \dfrac{K_1 K_2}{3} \dfrac{\sin\left[\dfrac{\pi}{3}-\dfrac{1}{3}\arccos\left(-K_2 \cos 3\theta\right)\right]}{\left(1-K_2^2\cos^2 3\theta\right)^{1/2}} \dfrac{\partial \cos 3\theta}{\partial t}; & \text{for } \cos 3\theta \leqslant 0 \end{cases} \tag{56}$$

where t denotes the loading parameter. Using the identity $\sin^2 x = 1-\cos^2 x$ we obtain by means of eqn (53)

$$\dot{\lambda} = \frac{\alpha_1}{\sqrt{3}} \frac{\partial \cos 3\theta}{\partial t}; \quad \text{for all } \cos 3\theta \text{ values} \tag{57}$$

where the nonnegative dimensionless quantity α_1 is defined by

$$\alpha_1 = \frac{K_1 K_2}{\sqrt{3}} \left[\frac{1-\left(\dfrac{\lambda}{K_1}\right)^2}{1-K_2^2\cos^2 3\theta} \right]^{1/2}. \tag{58}$$

It appears that α_1 is completely determined by the current stress state. Using eqn (54) we obtain

$$\dot{\lambda} = \sqrt{\left(\frac{2}{3}\right)} \alpha_1 \frac{\tau_0 \dot{J}_3 - 3J_3 \dot{\tau}_0}{\tau_0^4}. \tag{59}$$

Let us also define the following dimensionless quantity

$$\alpha_2 = A\frac{\tau_0}{\sigma_*} + \frac{1}{\sqrt{6}}\lambda - \alpha_1 \frac{J_3}{\tau_0^3}. \tag{60}$$

As σ_* for any stress state is determined so that eqn (55) is fulfilled, then α_2 is also determined by the current stress state.

Keeping the σ_*-value fixed, eqn (55) can now be differentiated and we obtain by means of eqns (40), (59), and (60)

$$\dot{\tau}_0 + \frac{B}{\alpha_2}\dot{\sigma}_0 + \frac{\alpha_1}{\alpha_2}\dot{J} = 0. \tag{61}$$

The objective of the derivations above is to choose the material moduli in such a way that the neutral deviatoric loading surface becomes affine with the failure surface in question. This is achieved by making eqns (50) and (61) identical. This implies the following restrictions on the material moduli

$$\frac{B_2^c B_1' + p_1\left(A_2' B_1' - A_1'^c B_2'^c\right)\left(1+\dfrac{A_3'}{A_2'}\dfrac{J_3}{\tau_0^3}\right)}{A_1^c B_2'^c - A_2' B_1' + A_1'^c B_2'^c} = \frac{B}{\alpha_2} \tag{62}$$

$$\frac{A_3'}{A_2'} = \frac{\alpha_1}{\alpha_2}. \tag{63}$$

Due to eqn (49), the last expression infers that $B_3'/B_2'^c = \alpha_1/\alpha_2$. Suppose that the θ-dependence is ignored, i.e. according to eqn (53) we have $K_2 = 0$ implying that $\alpha_1 = 0$. From

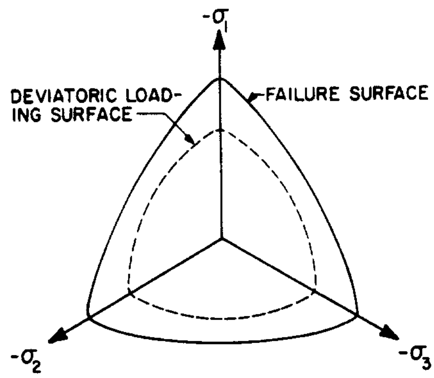

Fig. 3. Affinity between failure curve and neutral deviatoric loading curve in the deviatoric plane.

above we obtain $A'_3 = B'_3 = 0$, i.e. $A_3 = B_3 = 0$ in accordance with the previous discussion. Suppose instead that the failure surface is taken to be independent of the hydrostatic stress component, i.e. $B = 0$. It appears from eqn (62) that $A_1'^c = B'_1 = 0$ provides a solution; which, in turn, means that $A_1^c = 0$. However, these restrictions on the moduli infer that the response to purely hydrostatic loading becomes linear in accordance with general expectations for the response of pressure insensitive materials. Thus, the restrictions imposed by eqns (62) and (63) seem to be physically reasonable and, in addition, they imply that the neutral deviatoric loading surface becomes affine with the failure surface and at failure, the two surfaces coincide. In the deviatoric plane, this aspect is illustrated in Fig. 3.

As previously discussed, only the coefficient in front of $\dot{\sigma}_0$ differs for the two neutral loading surfaces given by eqns (50) and (51) and this coefficient determines the slope of the neutral curve in a meridian plane. For the neutral deviatoric surface the slope is affine to the slope of the failure curve as given by eqn (61). It appears that the slope of the neutral volumetric loading surface can be made positive by a proper choice of material functions, inferring that this surface takes the form of a "cap", a concept often applied in soil plasticity. This aspect as well as the affinity between the neutral deviatoric loading surface and the failure surface already have been anticipated in the layout of Fig. 2.

It appears that it is fairly simple to calibrate the material moduli so that advantage can be taken of the knowledge of a failure criterion. The fact that the present calibration also contains the Drucker–Prager and von Mises criteria makes the formulation quite appealing.

DIRECTION OF THE INCREMENTAL STRAIN VECTOR

In a further attempt to discuss some of the principal features of the proposed constitutive theory, it might be of interest to investigate the direction of the nonelastic, i.e. plastic part of the incremental strain vector. For this purpose it is convenient to consider the deviatoric and volumetric parts of the strain vector separately. From eqns (21) and (22) we obtain

$$\dot{e}^p_{ij} = \frac{\dot{L}_d - p_1\dot{L}_v}{B^c_2\tau_0} s_{ij} \tag{64}$$

$$\dot{\varepsilon}^p_0 = \frac{p_2\dot{L}_d - \dot{L}_v}{A^c_1} \tag{65}$$

where the superscript p refers to plasticity. From eqn (64) it appears that $\dot{e}^p_{ij}$ is colinear with s_{ij}. As the deviatoric loading surface, in general, has a triangular shape, cf. Fig. 3, it means that the model, in this respect, corresponds to nonassociated plasticity theory. However, the colinearity of $\dot{e}^p_{ij}$ and s_{ij} seems to be in perfect agreement with results obtained for concrete, cf. Fig. 4, and in reasonable agreement with results obtained for sand, cf. Fig. 5.

Moreover, in accordance with well-established evidence for frictional materials, it

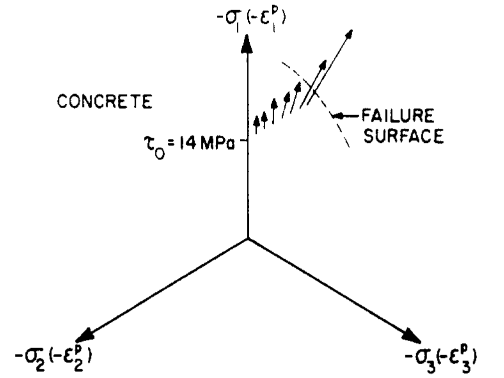

Fig. 4. Plastic strain directions for concrete for nonproportional loading in the deviatoric plane ($\sigma_0 = -28$ MPa), Stankowski and Gerstle[4].

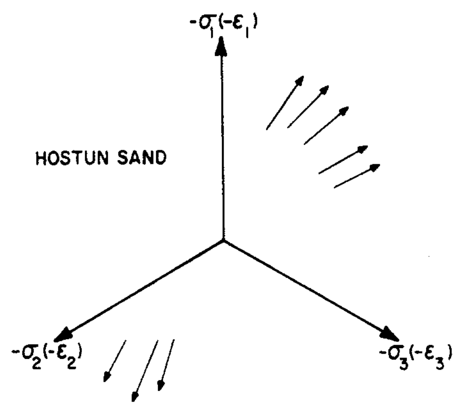

Fig. 5. Strain increments for Hostun sand at failure for proportional loading in the deviatoric plane ($\sigma_0 = -0.3$ MPa), Darve and Labanieh[17].

appears from eqn (65) that the plastic volumetric strain increment, in general, is also nonassociated with the loading surfaces. Consequently, the features demonstrated above in combination with the general properties of proposed theory seem to all correspond to observed behaviour of frictional materials. On this background, it is of interest to compare the present framework with some previous formulations. This comparison is performed below.

SIMPLIFICATIONS AND COMPARISONS WITH OTHER CONSTITUTIVE THEORIES

As outlined above, the present formulation seems to be quite general and to be able to make a comparison with previous theories, significant simplifications must be invoked.

Let us first assume that the loading surfaces degenerate into one and the same surface. This is achieved if $\dot{L}_d$ and $\dot{L}_v$ become proportional which, according to eqns (19) and (20), requires that $B'_1/A'^c_1 = B'^c_2/A'_2 = B'_3/A'_3$, i.e.

$$A'_2B'_1 - A'^c_1B'^c_2 = 0; \quad A'_3B'_1 - A'^c_1B'_3 = 0; \quad A'_2B'_3 - A'_3B'^c_2 = 0 \tag{66}$$

where we have already taken advantage of the last equation in the previous discussion, cf. eqn (49). Use of eqn (66) in expression (20) for $\dot{L}_v$ infers $\dot{L}_v = -A'_2\dot{L}_d/B'^c_2$ which, by means of eqns (21) and (22), yields

$$\dot{e}_{ij} = \frac{\dot{s}_{ij}}{B^c_2} + \frac{B^c_2\dot{L}_d}{B^c_2B'^c_2\tau_0}s_{ij} \tag{67}$$

$$\dot{\varepsilon}_0 = \frac{\dot{\sigma}_0}{A^c_1} + \frac{A_2\dot{L}_d}{A^c_1B'^c_2}. \tag{68}$$

These equations correspond to an isotropic hardening, nonassociated plasticity formulation with the yield surface ($\dot{L}_d = 0$) depending on all three stress invariants.

Let us now specialize one step further and assume that

$$A'_3 = B'_3 = 0 \tag{69}$$

which, according to eqns (17) and (18), infers that $A_3 = B_3 = 0$, i.e. the influence of the third stress invariant on the yield surface is neglected. Using eqns (66) and (69), it is easily shown that the determinant given by eqn (42) degenerates to

$$\det = B^c_2(A^c_1 B_2 + B^c_2 A^c_1) \tag{70}$$

and $\dot{L}_d$, as given by eqn (46), reduces to

$$\dot{L}_d = -\frac{B'_1 B^c_2 \dot{\sigma}_0 + A^c_1 B'^c_2 \dot{\tau}_0}{A^c_1 B_2 + B^c_2 A^c_1}. \tag{71}$$

Therefore, our yield surface now corresponds to a generalized Drucker–Prager surface depending only on σ_0 and τ_0. This, in turn, means that the constitutive equations, eqns (67) and (68), correspond to an isotropic hardening Drucker–Prager type of plasticity with associated plastic deviatoric strains and nonassociated plastic volumetric strain. In fact, the formulation given by eqns (67), (68), and (71) corresponds exactly to that proposed by Rudnicki and Rice[18] for rock and later applied to soil by Nemat-Nasser and Shokooh[19] and, essentially, also to concrete by Bazant and Kim[20]. Let us now investigate under which conditions this formulation reduces to an associated generalized Drucker–Prager plasticity theory. From eqns (67) and (68) we derive that the plastic strain rate is given by

$$\dot{\varepsilon}^p_{ij} = \frac{1}{B'^c_2}\left(\frac{B^c_2}{B^c_2 \tau_0} s_{ij} + \frac{A_2}{A^c_1}\delta_{ij}\right)\dot{L}_d \tag{72}$$

where $\dot{L}_d$ is expressed by eqn (71). For associated plasticity, the classical formulation is, cf. Hill[21]

$$\dot{\varepsilon}^p_{ij} = h\dot{f}\frac{\partial f}{\partial \sigma_{ij}} \tag{73}$$

where f is the yield surface and h is a hardening parameter. If eqns (72) and (73) are to be identical we must require

$$\frac{\partial f}{\partial \sigma_{ij}} = k\left(\frac{B^c_2}{B^c_2 \tau_0} s_{ij} + \frac{A_2}{A^c_1}\delta_{ij}\right); \qquad \dot{f} = \frac{1}{kB'^c_2 h}\dot{L}_d \tag{74}$$

where k is a proportionality factor. Using the expression for $\dot{L}_d$ given by eqn (71) as well as $\dot{f} = \dot{\sigma}_{ij}\,\partial f/\partial \sigma_{ij}$, the requirements above infer

$$\frac{A_2}{B^c_2} = \frac{B'_1}{B'^c_2} \tag{75}$$

i.e. we have shown that for this restriction on the material moduli, eqns (67), (68), and (71) correspond to an associated, generalized Drucker–Prager format.

A further simplification is obtained by assuming also that $A_2 = 0$ holds. According to eqn (75) we have $B'_1 = 0$, which, by means of eqn (66), infers that $A'^c_1 = 0$, i.e. $A^c_1 = B_1 = 0$.

Consequently, for

$$A_2 = 0, \text{ i.e. } B'_1 = A'^{c}_1 = 0 \quad \text{and} \quad A^{c}_1 = B_1 = 0 \tag{76}$$

eqn (71) becomes $\dot{L}_d = -B'^{c}_2\dot{\tau}_0/B_2$ and eqns (67) and (68) reduce to

$$\dot{e}_{ij} = \frac{\dot{s}_{ij}}{B^{e}_2} - \frac{B^{c}_2\dot{\tau}_0}{B^{e}_2 B_2 \tau_0} s_{ij} \tag{77}$$

$$\dot{\varepsilon}_0 = \frac{\dot{\sigma}_0}{A^{e}_1}. \tag{78}$$

Observing that only the deviatoric loading influences the plastic strains and that the volumetric behaviour is elastic, we have, in fact, achieved a formulation that is identical to the classical isotropic von Mises plasticity theory. Noting that the equivalent von Mises stress σ_e and the usual equivalent plastic strain rate $\dot{\varepsilon}^p_e$ are defined by $\sigma_e = 3\tau_0/\sqrt{2}$ and $\dot{\varepsilon}^p_e = (2\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij}/3)^{1/2}$, it appears readily that if H denotes the plastic modulus defined by $d\sigma_e/d\varepsilon^p_e = H$, then

$$B^{c}_2 = -\frac{B^{e^2}_2}{\frac{2}{3}H + B^{e}_2} = -\frac{E}{(1+\nu)\left[1 + \frac{2(1+\nu)}{3E}H\right]} \tag{79}$$

where the relation $B^{e}_2 = 2G = E/(1+\nu)$ has been used (E is Young's modulus and ν is Poisson's ratio).

The reduction to classical von Mises plasticity follows when $A_2 = A_3 = B_1 = B_3 = A^{c}_1 = 0$, i.e. the only remaining material moduli are $A_1 = A^{e}_1$ and $B_2 = B^{e}_2 + B^{c}_2$. With this interpretation in mind we can evaluate the present formulation in a slightly different light. In von Mises plasticity the nonlinearity is prescribed by a unique relation between the stress invariant σ_e and the strain invariant ε^p_e. In the present formulation we express instead von Mises plasticity by the invariant relation $\dot{\sigma}_0 = A^{e}_1\dot{\varepsilon}_0$ corresponding to linear volumetric behaviour and the invariant relation $\dot{\tau}_0 = (B^{e}_2 + B^{c}_2)\dot{e}$ describing the nonlinear deviatoric response. It is of considerable interest to observe that usually the von Mises plasticity theory relies on a decomposition of the total strain into elastic and plastic parts, whereas we here use an alternative description using *total* deviatoric strains by means of $\dot{e}$. With this in mind, the present general format can be viewed as a formulation where more and more invariants are included in the "effective" stress–strain relations depending on how complicated the actual material behaviour is, i.e. the present formulation seems to offer a quite natural extension of the basic ideas of conventional plasticity theory. However, an extremely important difference is that here we start directly with the stress–strain relations and then later add appropriate loading/unloading criteria, whereas the traditional plasticity approach is to envisage the size, shape and change of the yield surfaces. While this traditional approach is extremely helpful when dealing with materials having easily detectable yield surfaces, like mild steel, the present approach seems to be far more natural and advantageous when dealing with materials having gradually changing nonlinear behaviour over most of their loading range. In addition, the present formulation introduces loading and unloading criteria in a very simple and convenient manner; and these criteria are applicable also in the softening range. Moreover, contrary to normal plasticity theory where a decomposition of the total strain in elastic and plastic strains is necessary, the present formulation works always in terms of total strains. This aspect seems to be of significant importance for large strain analysis in which the decomposition into elastic and plastic strains is by no means obvious.

It follows from the previous discussion that if the loading/unloading criteria are dropped, the present formulation becomes one of hypoelasticity of third grade and as such it contains some formulations proposed previously for frictional materials. In particular, the soil models of Romano[22] and Davis and Mullenger[23] emerge if $A_3 = B_3 = 0$, and if $A_3 = B_3 = B_2^c = 0$ then the concrete model proposed by Coon and Evans[24] follows. Considering loading in the Rendulic plane and putting $A_3 = B_3 = A_2 = B_1 = 0$, we recover the variable moduli soil model of Nelson and Baron[25]. Moreover, the relations to the octahedral formulations proposed for concrete[2–4] have already been touched upon in the beginning of the paper.

Finally, if

$$A_3 = B_3 = A'_3 = B'_3 = B'_1 = B_1 = A'^c_1 = A^c_1 = 0 \tag{80}$$

then $\dot{L}_d = -B'^c_2 \dot{e}$ and $\dot{L}_v = A'_2 \dot{e}$. If A'_2 is allowed to vary, independently of whether volumetric loading or unloading occurs, this means that the volumetric loading criterion is deactivated and only the deviatoric loading criterion is of importance. Therefore, B'^c_2 becomes the only material modulus that is influenced by loading and unloading. If B^c_2 is taken as a constant whereas A^c_1 is allowed to vary, then the present formulation by a proper choice of the material functions degenerates to the elastic-fracturing model proposed very recently by Resende and Martin[1] for concrete and rock.

Consequently, the present theory seems to be of surprising generality as it contains formulations ranging from nonassociated plasticity theory, associated theory, hypoelasticity to elastic-fracturing theory. Even so, the derivation of the theory is rather straightforward.

CONCLUSIONS

A new theoretical framework for constitutive modelling of frictional materials has been proposed. The basis of the theory is two very general relations between appropriately defined stress and strain rate invariants. These invariant equations are then expanded so that the relations between all the components of the stress and strain rate tensors are established. This implies a linear relation between the stress rate tensor and the strain rate tensor defining the tangential stiffness tensor. The tangential stiffness tensor can be used directly in numerical applications, but it is shown, in general, to be nonsymmetrical.

The established incremental stress–strain relations are then augmented by appropriate loading/unloading criteria and in accordance with the general behaviour of frictional materials, these criteria include deviatoric as well as volumetric loading/unloading rules. The proposed criteria are consistent in the sense that they fulfil the important continuity requirement. Moreover, it was demonstrated that the deviatoric loading surface can easily be calibrated to be affine with failure surfaces that depend on all three stress invariants.

Emphasis was given to a fundamental discussion of the general properties of the proposed theory and it was shown that the theory fulfils all the formal requirements that a properly formulated constitutive theory must obey. Despite the rather straightforward derivation, it was also shown that the theory possesses the potential to model all the important features of frictional material behaviour such as: influence of all three stress invariants, coupling between deviatoric and volumetric response, dilatancy, softening and different behaviour in loading and unloading.

Moreover, the proposed theory was demonstrated to contain a surprisingly large number of both classical and nonclassical theories as special cases. In particular, it contains formulations ranging from nonassociated plasticity theory, associated plasticity theory, hypoelasticity to elastic-fracturing theory.

Finally, it is of considerable interest that the proposed theory does not rely upon a decomposition of the total strains into elastic and plastic parts. Even though only small strains have been considered in the present paper, this aspect seems to offer great advantages when also kinematic nonlinearities are involved.

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